

Varying Interest Rates and Annuities

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Abstract

In this paper, we first applied a piecewise linear regression model to best fit the discrete varying interest rate data and then we applied these results for the calculation of annuities with time variable interest rates. We used two different approaches to calculate annuities with time-varying interest rates, namely, the Yield Curve Method and the Portfolio Rate Method.

Keywords: Annuity-immediate, Piecewise Linear Regression, Yield Curve Method, Portfolio Rate Method, Varying Interest Rates.

1 Introduction

The purpose of this paper is to fill the gap in evaluations of annuities for piecewise time variable interest rates in the spot rate method and the forward rate method. In the literature, only piecewise constant interest rates have been applied to these two methods. Also many results are available in literature where interest rates considered are random time variables following Brownian or Levy motions. The project intends to consider the cases in between the simplistic cases and the sophisticated cases.

This paper researches the intersection of economics and mathematics and it does that through the lens of varying interest rates and their effects on annuity calculations, which are undoubtedly crucial for financial planning and risk assessment. It introduces piecewise linear regression models to improve traditional constant rate assumptions, which provide a more precise method for modeling financial products affected by economic instabilities. This advancement displays the need for accurate financial modeling due to the rapidly increasing complexity and interconnectedness of global markets. By improving the understanding of annuity behavior under realistic conditions, this research will offer valuable insights for policymakers and financial professionals. Which consequently will aid the development of efficient economic strategies and investment decisions.

2 Annuities with Fixed and Variable Interest Rates

In this section, we will explain what is annuity and how it is calculated when the interest rate does not change, and when it changes.

2.1 Fixed Interest Rate

Definition 1.1.1. An *annuity* is a series of payments made at equal intervals [3].

Definition 1.1.2. An *interest rate* is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period [3].

Definition 1.1.3. The *annuity-immediate* is an annuity when the payment is made at the end of each period for n periods [3].

Usually, for simplicity, we assume that the interest rate is constant for all conversion periods when we calculate annuities. For the context of this paper, we will only discuss annuity-immediate and assume that the payment at each interval is equal to 1.

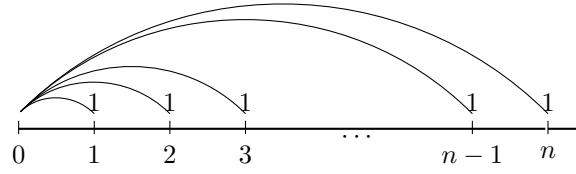


Figure 1: Time-diagram for an annuity-immediate

We use this formula to find the present value of such annuity-immediate [3]:

$$a_{\overline{n}|} = \frac{1 - \nu^n}{i} \quad (1)$$

$$\nu = \frac{1}{1 + i} \quad (2)$$

where $a_{\overline{n}|}$ is the present value of the annuity, i is a constant interest rate for a given period, and n is the number of the interest conversion periods.

However, in the real world interest rates change regularly in time, and the formula (1) is not accurate in this case. According to the data from the World Bank (Figure 2), the interest rate is constantly changing because of political, economic, and environmental factors.

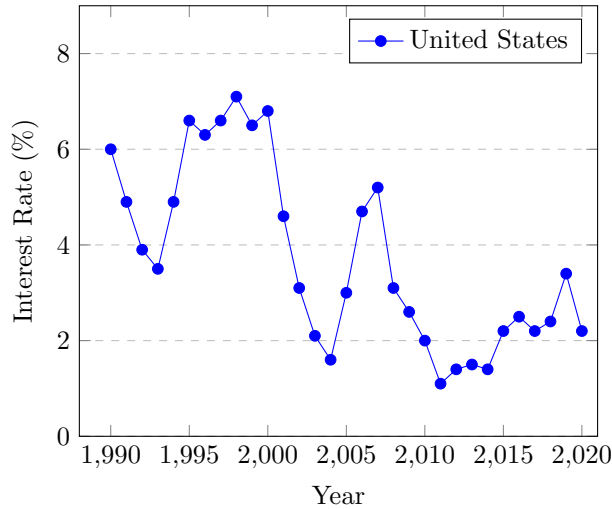


Figure 2: Real Interest Rate (%) - United States [8]

In real life, we cannot find annuity exactly using the formula (1). So, we need to use a different approach.

2.2 Time Varying Interest Rate

We can use two different methods to find the present value of the annuity with discrete time-varying interest rates.

The first approach is the Yield Curve Method [3]:

$$a_{\overline{n}|} = (1 + i_1)^{-1} + (1 + i_2)^{-2} + \dots + (1 + i_n)^{-n} = \sum_{t=1}^n (1 + i_t)^{-t} \quad (3)$$

where i_t is the interest rate at time t .

The idea behind this method is that we use the interest rate i_t for the payment made at time t overall t periods [3].

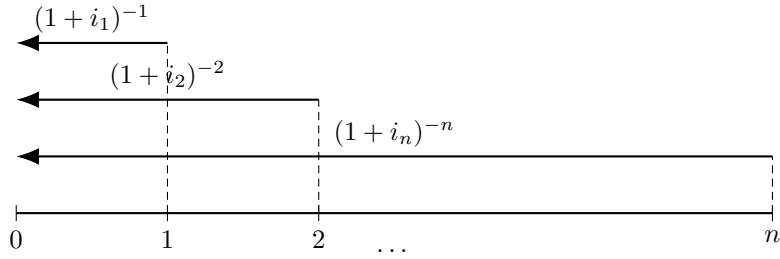


Figure 3: Time-diagram for the yield curve method

The second approach is the Portfolio Rate Method [3]:

$$a_{\overline{n}|} = (1 + i_1)^{-1} + (1 + i_1)^{-1}(1 + i_2)^{-1} + \dots + (1 + i_1)^{-1}(1 + i_2)^{-1} \dots (1 + i_n)^{-1} = \sum_{t=1}^n \prod_{s=1}^t (1 + i_s)^{-1} \quad (4)$$

where i_t is the interest rate at time t .

In this method, the interest rate i_t is applicable only for period t regardless of when the payment is made [3].

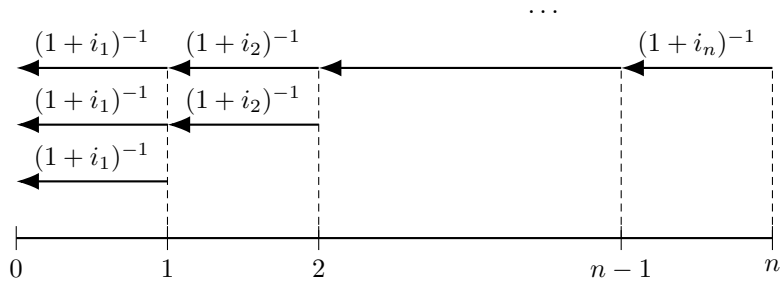


Figure 4: Time-diagram for the portfolio rate method

3 Piecewise Linear Regression

Before we continue with the topic of annuities, we need to introduce the topic of piecewise linear regression and how we can use a piecewise linear regression model for time-varying interest rate data for the calculation of the annuities.

3.1 Introduction to the Theory of Piecewise Linear Regression

The idea that lies behind the *piecewise linear regressions* is that if the data follow different linear trends over different regions of the data then we should model the regression function in "pieces" [6].

The basic formula for the piecewise linear regression [7]:

$$y(x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \epsilon \quad (5)$$

where $x_1, x_2, \dots, x_{n-1}, x_n$ are the cut-off points where the regression line changes, and β_{n+1} represents the slope of the line after each cut-off point, ϵ is the error term.

For a piecewise linear regression model, the equation is modified to include different regression lines for different regions of data of some independent variable. This can be expressed as:

$$y(x) = \begin{cases} \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \epsilon, & \text{if } x \leq x_1 \\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \beta_{n+1}(x - x_1) + \epsilon, & \text{if } x_1 < x \leq x_2 \\ \dots & \\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \beta_{n+1}(x - x_{n-1}) + \epsilon, & \text{if } x_{n-1} < x \leq x_n \\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \beta_{n+1}(x - x_n) + \epsilon, & \text{if } x > x_n \end{cases} \quad (6)$$

where x_1, x_2, \dots, x_n are the cut-off points where the regression line changes, and β_{n+1} represents the slope of the line after each cut-off point.

3.2 Piecewise Linear Interest Rate Function

Now, with the data for time-varying interest rate and formula (6), we can construct a piecewise linear regression function for $i(t)$. For $k = 0, 1, 2, \dots, n - 1, n$ the time-varying interest rate function $i_k(t)$ from $t = k$ to $t = k + 1$ where the cut-off points are the values of interest rate at each region:

$$i_k(t) = a_k \cdot t + b_k \quad (7)$$

where $i := y, t := x$, $a_k = \beta_1, \beta_2, \beta_3, \dots, \beta_n, \beta_n + 1$, is the slope, and b_k is some constant that define i -intersection.

Using the data from the World Bank (Figure 2) and formula (7) (see Python code in Appendix), we found values for a_k and b_k for the interest rate data (Note: for all further linear regression calculations, we assume our zero point is 1990 year with the 6% interest rate).

4 Numerical Integration of Annuity with Time-Varying Interest Rate

4.1 Yield Curve Method

In this method, we will not use directly the values of the interest rates from Table 1, but rather we will assume that the interest rate at time $t = k$ is the average of the interest rates from time $t = 0$

Date	Interest Rate	Date	Interest Rate	Date	Interest Rate
1991	4.9%	2001	4.6%	2011	1.1%
1992	3.9%	2002	3.1%	2012	1.4%
1993	3.5%	2003	2.1%	2013	1.5%
1994	4.9%	2004	1.6%	2014	1.4%
1995	6.6%	2005	3.0%	2015	2.2%
1996	6.3%	2006	4.7%	2016	2.5%
1997	6.6%	2007	5.2%	2017	2.2%
1998	7.1%	2008	3.1%	2018	2.4%
1999	6.5%	2009	2.6%	2019	3.4%
2000	6.8%	2010	2.0%	2020	2.2%

Table 1: Real Interest Rate (%) - United States

k	a_k, b_k	k	a_k, b_k	k	a_k, b_k
0	-0.011, 0.06	10	-0.022, 0.288	20	-0.009, 0.20
1	-0.01, 0.059	11	-0.015, 0.211	21	0.003, -0.052
2	-0.004, 0.047	12	-0.01, 0.151	22	0.001, -0.008
3	0.014, -0.007	13	-0.005, 0.086	23	-0.001, 0.038
4	0.017, -0.019	14	0.014, -0.18	24	0.008, -0.178
5	-0.003, 0.081	15	0.017, -0.225	25	0.003, -0.053
6	0.003, 0.045	16	0.005, -0.033	26	-0.003, 0.103
7	0.005, 0.031	17	-0.021, 0.409	27	0.002, -0.032
8	-0.006, 0.119	18	-0.005, 0.121	28	0.01, -0.256
9	0.003, 0.038	19	-0.006, 0.14	29	-0.012, 0.382

Table 2: Linear Models for Interest Rate

to $t = k$:

$$\bar{i}_k = \frac{\sum_{k=1}^n i_k}{k} \quad (8)$$

for $k = 1, 2, \dots, n$.

In Table 3 are the average values of the interest rates.

k	Average Interest Rate	k	Average Interest Rate	k	Average Interest Rate
1	4.9	11	5.61	21	4.3
2	4.4	12	5.4	22	4.16
3	4.1	13	5.15	23	4.05
4	4.3	14	4.89	24	3.94
5	4.476	15	4.77	25	3.87
6	5.02	16	4.76	26	3.82
7	5.24	17	4.79	27	3.76
8	5.48	18	4.69	28	3.71
9	5.59	19	4.58	29	3.7
10	5.71	20	4.46	30	3.65

Table 3: Average Interest Rate from $t = 0$ to $t = k$

With values from Table 3 and piecewise linear regression model (see Python code in Appendix), we found a_k and b_k for the average interest rates from time $t = 0$ to $t = k$:

Now, using the piecewise linear regression model and our assumption about the average interest rate, we can rewrite formula (3):

k	a_k, b_k	k	a_k, b_k	k	a_k, b_k
0	-0.011, 0.06	10	-0.001, 0.0671	20	-0.0016, 0.0766
1	-0.005, 0.054	11	-0.0021, 0.0792	21	-0.0014, 0.0724
2	-0.003, 0.05	12	-0.0025, 0.084	22	-0.0012, 0.0658
3	0,002, 0.035	13	-0.0026, 0.0853	23	-0.0011, 0.0658
4	0.0046, 0.0246	14	-0.0012, 0.0657	24	-0.0011, 0.0562
5	0.0026, 0.0346	15	4.039E-18, 0.0493	25	-0.0007, 0.0512
6	0.0022, 0.037	16	0.0003, 0.0428	26	-0.0005, 0.0538
7	0.0024, 0.0356	17	-0.001, 0.0649	27	-0.0006, 0.0511
8	0.0011, 0.046	18	-0.0011, 0.0667	28	-1E-04, 0.0399
9	0.0012, 0.0451	19	-0.0012, 0.0686	29	-0.0005, 0.0515

Table 4: Figure 5: Linear Model for Average Interest Rate

$$a_{\overline{n}|t} = \sum_{k=0}^{n-1} \int_k^{k+1} \frac{1}{(a_k t + b_k + 1)^{k+1}} dt \quad (9)$$

Now, let's simplify this formula. First, let's find the value of integral from k to $k + 1$:

$$\int_k^{k+1} \frac{1}{(a_k t + b_k + 1)^{k+1}} dt = \frac{(1 + b_k + a_k t)^{-k}}{-a_k k} \Big|_k^{k+1} = \frac{(1 + b_k + a_k(k+1))^{-k}}{-a_k k} - \frac{(1 + b_k + a_k k)^{-k}}{-a_k k} \quad (10)$$

works for all $k \neq 0$.

Then, let's rewrite formula (10) such that we take out a term with $k = 0$:

$$a_{\overline{n}|t} = \int_0^1 \frac{1}{a_0 t + b_0 + 1} dt + \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{(a_k t + b_k)^{k+1}} dt \quad (11)$$

Calculating integrals of two terms separately, we get:

$$a_{\overline{n}|t} = \left(\frac{1}{a_0} \ln|a_0 t + b_0 + 1| \right) \Big|_0^1 + \sum_{k=1}^{n-1} \left(\frac{(1 + b_k + a_k(k+1))^{-k}}{-a_k k} - \frac{(1 + b_k + a_k k)^{-k}}{-a_k k} \right) \quad (12)$$

$$a_{\overline{n}|t} = \frac{1}{a_0} (\ln|a_0 + b_0 + 1| - \ln|b_0 + 1|) + \sum_{k=1}^{n-1} \left(\frac{(1 + b_k + a_k(k+1))^{-k}}{-a_k k} - \frac{(1 + b_k + a_k k)^{-k}}{-a_k k} \right) \quad (13)$$

Now, we will compare the results by using formulas (3) and (13). Firstly, we calculate directly the present value of the annuity using formula (3) and values from Table 3 for $n = 10, n = 20, n = 30$:

$$a_{\overline{10}|t} = \sum_{t=1}^{10} (1 + i_t)^{-t} = 7.6789413$$

$$a_{\overline{20}|t} = \sum_{t=1}^{20} (1 + i_t)^{-t} = 12.499443$$

$$a_{\overline{30}|t} = \sum_{t=1}^{30} (1 + i_t)^{-t} = 16.301147$$

Next, we calculated annuities using data from Table 4 and formula (13):

$$a_{\overline{10}|t} = \frac{1}{a_0} (\ln|a_0 + b_0 + 1| - \ln|b_0 + 1|) + \sum_{k=1}^9 \left(\frac{(1 + b_k + a_k(k+1))^{-k}}{-a_k k} - \frac{(1 + b_k + a_k k)^{-k}}{-a_k k} \right) = 7.7006641$$

$$a_{\overline{20}|t} = \frac{1}{a_0} (\ln|a_0 + b_0 + 1| - \ln|b_0 + 1|) + \sum_{k=1}^{19} \left(\frac{(1 + b_k + a_k(k+1))^{-k}}{-a_k k} - \frac{(1 + b_k + a_k k)^{-k}}{-a_k k} \right) = 12.004191$$

$$a_{\overline{30}|t} = \frac{1}{a_0} (\ln|a_0 + b_0 + 1| - \ln|b_0 + 1|) + \sum_{k=1}^{29} \left(\frac{(1 + b_k + a_k(k+1))^{-k}}{-a_k k} - \frac{(1 + b_k + a_k k)^{-k}}{-a_k k} \right) = 15.769703$$

4.2 Portfolio Rate Method

Using piecewise linear regression, we modify formula (4):

$$\begin{aligned} a_{\overline{n}|t} &= \int_0^1 (1 + i_0(t))^{-1} dt + \int_0^1 (1 + i_0(t))^{-1} dt \int_1^2 (1 + i_1(t))^{-1} dt + \dots + \int_0^1 (1 + i_0(t))^{-1} dt \int_1^2 (1 + \\ &\quad + i_1(t))^{-1} dt \dots \int_{n-1}^n (1 + i_{n-1}(t))^{-1} dt \\ a_{\overline{n}|t} &= \sum_{t=1}^{n-1} \prod_{k=0}^t \int_k^{k+1} \frac{1}{a_k t + b_k + 1} dt \end{aligned} \quad (14)$$

Next, we integrate the formula (14):

$$a_{\overline{n}|t} = \sum_{t=1}^{n-1} \prod_{k=0}^t \left(\frac{1}{a_k} \ln|a_k t + b_k + 1| \right) \Big|_k^{k+1} = \sum_{t=1}^n \prod_{k=0}^t \frac{1}{a_k} (\ln|a_k(k+1) + b_k + 1| - \ln|a_k k + b_k + 1|) \quad (15)$$

To calculate annuity and check formula (15), we will directly use values from Table 1 and Table 2. At first, we calculated annuity using formula (4) when the interest rates are discrete values and next, we calculated annuity using formula (15) when the interest rate is a linear function of time for $n = 10, 20, 30$.

$$a_{\overline{10}|t} = \sum_{t=1}^{10} \prod_{s=1}^t (1 + i_s)^{-1} = 7.68131072$$

$$a_{\overline{20}|t} = \sum_{t=1}^{20} \prod_{s=1}^t (1 + i_s)^{-1} = 12.5108512$$

$$a_{\overline{30}|t} = \sum_{t=1}^{30} \prod_{s=1}^t (1 + i_s)^{-1} = 16.3313251$$

$$a_{\overline{10}|t} = \sum_{t=1}^9 \prod_{k=0}^t \frac{1}{a_k} (\ln|a_k(k+1) + b_k + 1| - \ln|a_k k + b_k + 1|) = 7.7000138$$

$$a_{\overline{20}|t} = \sum_{t=1}^{19} \prod_{k=0}^t \frac{1}{a_k} (\ln|a_k(k+1) + b_k + 1| - \ln|a_k k + b_k + 1|) = 12.5987304$$

$$a_{\overline{30}|t} = \sum_{t=1}^{29} \prod_{k=0}^t \frac{1}{a_k} (\ln|a_k(k+1) + b_k + 1| - \ln|a_k k + b_k + 1|) = 16.4981717$$

5 Conclusion

This paper has shed light on the challenges of dealing with annuities with varying interest rates, a real-world issue affected by various factors. Through our calculations and application of the piecewise linear regression model, we derived two formulas for annuities with time-varying interest rates using Yield Curve and Portfolio Rate Methods.

	Yield Curve Method		
	Formula 3	Formula 13	Difference in values
n=10	7.6789413	7.7006641	-0.0217228
n=20	12.499443	12.004191	0.495252
n=30	16.301147	15.769703	0.531444
	Portfolio Rate Method		
	Formula 4	Formula 15	Difference in values
n=10	7.68131072	7.7000138	-0.01870308
n=20	12.5108512	12.5987304	-0.0878792
n=30	16.3313251	16.4981717	-0.1668466

Table 5: Comparison of results for two methods using different formulas

Table 5 shows the results that we got. According to Table 5, the difference is relatively small between using two methods with discrete interest values and with piecewise linear function for interest rate. However, the Portfolio Rate Method is more accurate because when we use it we take into account the fact that the interest rate is different for each period, while in the Yield Curve Method, we neglect this fact.

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A Appendix

Python Code for Linear Regression for the Interest Rate

```
def find_line_segments(points):
    results = []
    for i in range(len(points) - 1):
        x1, y1 = points[i]
        x2, y2 = points[i + 1]

        # Calculate the slope (m)
        if x2 - x1 != 0:
            m = (y2 - y1) / (x2 - x1)
        else:
            results.append("The line is vertical, so the slope is undefined.")
            continue

        # Calculate the y-intercept (b)
        b = y1 - m * x1

        results.append((m, b))

    return results

# Manually define each point
point_1 = (1, 3)
point_2 = (2, 4)
point_3 = (3, 7)
# Add more points up to point_30
point_30 = (30, 60) # Example: last point

# List of all points
points = [
    point_1, point_2, point_3,
    # Include all intermediate points here
    point_30
]

# Calculate line segments
segment_details = find_line_segments(points)

# Printing results
for index, details in enumerate(segment_details, 1):
    if isinstance(details, tuple):
        print(f"Segment {index}: Slope = {details[0]}, Intercept = {details[1]}")
    else:
        print(f"Segment {index}: {details}")
```